Perfectly Matched Layer Absorbing Boundary Condition for Dispersive Medium

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***Abstract*—Berenger's perfectly matched layer (PML) absorbing boundary condition (ABC) has been found very effective for truncating the unbounded spatial domain in the finite-difference time-domain (FDTD) computation. The PML ABC was originally introduced for a free-space spatial domain and later extended to a lossy medium using the stretched coordinates. In this paper, we propose a novel PML ABC for a dispersive medium in an ordinary Cartesian coordinate. It will be also shown that the PML for the lossy medium can be easily derived from our formulation.**

***Index Terms*—Absorbing boundary condition, dispersive medium, FDTD, perfectly matched layer.**

I. Introduction

THE PERFECTLY matched layer (PML), proposed by Berenger, behaves as an excellent absorber in the finite-difference time-domain (FDTD) computation for electromagnetic scattering and radiation problems [1], [2]. The PML concept was first introduced for free-space or lossless medium. It has been demonstrated in many numerical applications that the PML can absorb the outgoing propagating waves very effectively, but cannot effectively absorb the evanescent waves [3]. Fang and Wu modified the original PML in order to apply to a lossy medium [4], [5] in which the stretched coordinates [6] were used. Gedney extended a uniaxial PML [7] to lossy and dispersive media [8]. It has been shown that these PML's can effectively absorb both propagating and evanescent waves.

In this paper, a novel ABC for the dispersive medium is derived in the ordinary Cartesian coordinate, based on the concept of Berenger's PML. First, we derive the PML ABC's for Debye and Lorentz dispersions. These ABC's are extended to a more general case. It is also shown that the PML for lossy medium can be easily derived from our PML formulation as a special case. While our discussion will be restricted to spatially uniform frequency-dependent permittivities, frequency-dependent magnetic materials can be also treated by using the same approach described in this letter.

II. Matching Condition for Dispersive Medium

Since similar results can be obtained for either two- as well as three-dimensional cases, we consider the two-dimensional TE-case with field components E_x, E_y, and H_z. As is analogized from the conventional PML equations for nondispersive lossless medium, the displacement current terms $\varepsilon\partial\mathbf{E}/\partial t$ should be replaced by $\partial\mathbf{D}/\partial t$, and then the PML equations for the dispersive medium are written as follows:

$$\frac{\partial D_x}{\partial t} + \sigma_{dy}E_x = \frac{\partial(H_{zx}+H_{zy})}{\partial y} \tag{1}$$

$$\frac{\partial D_y}{\partial t} + \sigma_{dx}E_y = -\frac{\partial(H_{zx}+H_{zy})}{\partial x} \tag{2}$$

$$\mu\frac{\partial H_{zx}}{\partial t} + \sigma^*_{dx}H_{zx} = -\frac{\partial E_y}{\partial x} \tag{3}$$

$$\mu\frac{\partial H_{zy}}{\partial t} + \sigma^*_{dy}H_{zy} = \frac{\partial E_x}{\partial y} \tag{4}$$

where

$$\mathbf{D}(\omega,\mathbf{r}) = \varepsilon_d(\omega)\mathbf{E}(\omega,\mathbf{r}) \tag{5}$$

and $\varepsilon_d(\omega)$ is the complex permittivity of the dispersive PML medium. The equivalent time-domain relation can be represented as a convolution integral.

While our discussion is concentrated on the case that only the permittivity of the FDTD space has frequency dependence, the similar results will be obtained for the case that μ has frequency dependence by replacing the first terms of (3) and (4) with $\partial B_{zx}/\partial t$ and $\partial B_{zy}/\partial t$, respectively.

Following the approach in [1], a wave admittance Y_d of this PML medium is given by

$$Y_d = \sqrt{Y_{dx}^2\cos^2\phi + Y_{dy}^2\sin^2\phi} \tag{6}$$

where, for $i = x$ and y,

$$Y_{di} = \sqrt{\frac{\varepsilon_d(\omega) + \frac{1}{j\omega}\sigma_{di}}{\mu + \frac{1}{j\omega}\sigma^*_{di}}}. \tag{7}$$

Therefore, the impedance matching conditions of the dispersive PML for the medium whose permittivity and permeability are $\varepsilon(\omega)$ and μ, are given by

$$\frac{\mu + \frac{1}{j\omega}\sigma^*_{di}}{\varepsilon_d(\omega) + \frac{1}{j\omega}\sigma_{di}} = \frac{\mu}{\varepsilon(\omega)}, \quad i = x, y. \tag{8}$$

From (8) the permittivity ε_d in the dispersive PML is a function of the conductivities σ_{di} or σ^*_{di}, therefore, the flux density (5) is changed as $D_i = \varepsilon_{di}E_i$, for $i = x$ and y. However, the subscriptions x and y will be omitted in the following discussions because the same conditions for both x and y are satisfied.

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III. Single-Order Pole Dispersion

In this section, we will derive the PML ABC for the dispersive media whose complex permittivities are all described by the equations with the first-order poles. These dispersions can be used to model a wide variety of materials and are easily analyzed by using the recursive convolution scheme [9].

A. Debye Dispersion

The permittivity of the Debye material is describe by the following Debye equation:

$$\varepsilon(\omega) = \varepsilon_0\left\{\varepsilon_\infty + (\varepsilon_s - \varepsilon_\infty)\frac{1}{1 + j\omega/\omega_p}\right\} \tag{9}$$

where ε_s is the static permittivity at $\omega = 0$, ε_∞ is the infinite frequency permittivity, and $\tau_p = 1/\omega_p$ is the relaxation time.

In order to derive the dispersive PML medium, we assume the permittivity in the PML in the form

$$\varepsilon_d(\omega) = \varepsilon_0\left(\varepsilon_\infty + \frac{a_d}{1 + j\omega/\omega_p}\right). \tag{10}$$

Substituting (10) in (8), we obtain the following relations:

$$a_d = (\varepsilon_s - \varepsilon_\infty)\left(1 - \frac{1}{\omega_p}\frac{\sigma_d}{\varepsilon_0\varepsilon_s}\right) \tag{11}$$

$$\frac{\sigma_d}{\varepsilon_0\varepsilon_s} = \frac{\sigma_d^*}{\mu}. \tag{12}$$

B. Lorentz Dispersion

The permittivity of the Lorentz dispersion is described as

$$\varepsilon(\omega) = \varepsilon_0\left\{\varepsilon_\infty + \frac{(\varepsilon_s - \varepsilon_\infty)\omega_p^2}{\omega_p^2 + 2j\omega_p\delta_p - \omega^2}\right\} \tag{13}$$

where ω_p is the resonant frequency and δ_p the damping coefficient. Letting $j\omega = s$, and expanding the above equation into a partial fraction, we obtain

$$\varepsilon(s) = \varepsilon_0\left\{\varepsilon_\infty + \frac{(\varepsilon_s - \varepsilon_\infty)\omega_p^2}{s_1 - s_2}\left(\frac{1}{s - s_1} - \frac{1}{s - s_2}\right)\right\} \tag{14}$$

where s_1 and s_2 are two complex conjugate poles of (13), and we assume that $s_1 \neq s_2 \neq 0$ for simplicity.

Comparing (14) and (9) we find that the same discussion as the Debye dispersion can be applied, and we obtain the dispersive PML for Lorentz dispersion as follows:

$$\varepsilon_d(s) = \varepsilon_0\left(\varepsilon_\infty + \frac{a_1}{s - s_1} - \frac{a_2}{s - s_2}\right) \tag{15}$$

where

$$a_n = (\varepsilon_s - \varepsilon_\infty)\left(1 + \frac{1}{s_n}\frac{\sigma_d}{\varepsilon_0\varepsilon_s}\right) \qquad (n = 1, 2). \tag{16}$$

The relationship with respect to the electric and magnetic conductivities is the same as (12).

C. Multipole Dispersion

The above discussion can be easily extended to a more general case. We consider here the following dispersion:

$$\varepsilon(s) = \varepsilon_0\left\{\varepsilon_\infty + (\varepsilon_s - \varepsilon_\infty)\sum_{n=1}^{N}\frac{G_n}{s - s_n}\right\} \tag{17}$$

with the condition that $\sum_{n=1}^{N} G_n/s_n = -1$, and $s_n \neq 0$ for all n. In this case, we can obtain the following dispersive PML:

$$\varepsilon_d(s) = \varepsilon_0\left\{\varepsilon_\infty + \sum_{n=1}^{N}\frac{a_n}{s - s_n}\right\} \tag{18}$$

where

$$a_n = (\varepsilon_s - \varepsilon_\infty)\left(1 + \frac{1}{s_n}\frac{\sigma_d}{\varepsilon_0\varepsilon_s}\right)G_n \tag{19}$$

and the relation with respect to the electric and magnetic conductivities is again the same as (12).

D. Special Case

Up to this point the permittivity of the FDTD space has been described by a first order pole. However, the above discussion cannot be applied when the conductivity term that is equivalent to an additional first-order pole at $s = 0$ exists, because the coefficient a_n becomes infinity. Therefore we consider here the following dispersion:

$$\varepsilon(s) = \varepsilon_0\left\{\varepsilon_\infty + (\varepsilon_s - \varepsilon_\infty)\sum_{n=1}^{N}\frac{G_n}{s - s_n}\right\} + \frac{\sigma}{s}. \tag{20}$$

In this case the higher order term must be added to the permittivity of dispersive PML because the matching condition (8) is not satisfied. Then we let

$$\varepsilon_d(s) = \varepsilon_0\left\{\varepsilon_\infty + (\varepsilon_s - \varepsilon_\infty)\sum_{n=1}^{N}\frac{a_n}{s - s_n}\right\} + \frac{\sigma}{s} + \frac{b_0}{s^2} \tag{21}$$

and substituting (21) in (8), we obtain

$$b_0 = \frac{\sigma\sigma_d^*}{\mu} = \frac{\sigma\sigma_d}{\varepsilon_0\varepsilon_s}. \tag{22}$$

The coefficient a_n is given by (19), and the relation between σ_d and σ_d^* is the same as (12).

If we let $\varepsilon_s = \varepsilon_\infty$ as a special case, that is the FDTD space is filled with the lossless permittivity and conductivity terms, then we can easily obtain the PML medium as follows:

$$\varepsilon_d(s) = \varepsilon_0\varepsilon_s + \frac{\sigma}{s} + \frac{1}{s^2}\frac{\sigma\sigma_d}{\varepsilon_0\varepsilon_s}. \tag{23}$$

Substituting (23) in the equivalent frequency domain equation of (1) we obtain

$$-\frac{\partial(H_{zx} + H_{zy})}{\partial y} = \varepsilon_0\varepsilon_s s E_y + (\sigma + \sigma_{dy})E_y + \frac{\sigma\sigma_{dx}}{\varepsilon_0\varepsilon_s}\frac{1}{s}E_y. \tag{24}$$

This is very similar to the time domain field equation for the Generalized PML discussed in [4].

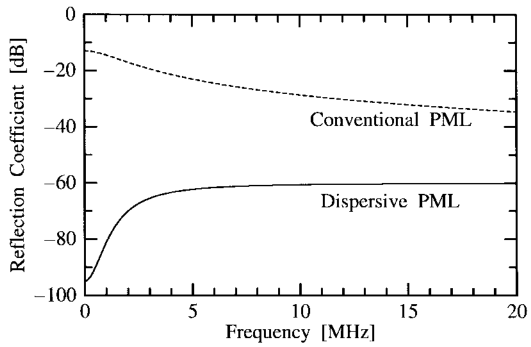

Fig. 1. Theoretical reflection coefficients of conventional and dispersive PML's. The parameters are $M = 2$, $L\Delta x = 4 \times 10^{-4}$, $\omega_p = 2\pi \times 10^6$, $\varepsilon_s = 10$, and $\varepsilon_\infty = 4$. The required reflection coefficient is set at $|R| = -60$ [dB]. The effective permittivity is $\varepsilon_e = \varepsilon_\infty$.

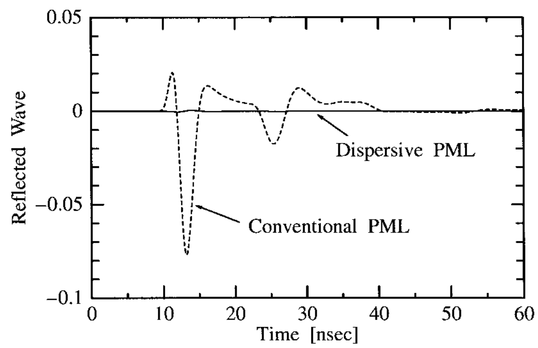

Fig. 2. Reflected waves at a corner of FDTD space. The parameters are $M = 2$, $|R| = -200$ [dB], $\varepsilon_s = 9$, $\varepsilon_\infty = 4$, $\varepsilon_e = \varepsilon_\infty$, $\omega_p = 2\pi \times 10^9$, and $\Delta x = \Delta y = 10$ mm.

IV. Reflection from the PML

Following [1], the reflection coefficient at the PML boundary can be easily obtained. When the conductivity profile in the PML is chosen as $\sigma_d(x) = \sigma_{\max}(\frac{x}{L\Delta x})^M$, the resultant reflection coefficient of L-layered PML backed by the perfect conductor for plane wave at normal incidence is approximated by

$$R = e^{-\frac{2}{v(\omega)}\left(j\omega + \frac{\sigma_{\max}}{\varepsilon_0 \varepsilon_s}\frac{1}{M+1}\right)L\Delta x} \qquad (25)$$

where $v(\omega)$ is the velocity in the FDTD space.

The value of $\sigma_{\max}$ cannot be determined from above equation because the absolute of R depends on the frequency as yet. In all numerical examples shown here, $\sigma_{\max}$ was determined from $|R|$ by using a frequency-independent effective permittivity ε_e.

Fig. 1 shows the theoretical reflection coefficients for conventional and dispersive PML media. The dispersion is Debye type. Fig. 2 shows the reflected waves at a corner of FDTD space when the magnetic line current is located at the center of 200×200 computation space including the 16-layered PML's. It is found that the significant absorption can be achieved. While not shown here, the significant improvement is obtained for other dispersions.

V. Conclusion

The PML ABC for the spatially uniform dispersive medium, which is an extension of the original Berenger's PML ABC, has been proposed. The PML ABC's for Debye and Lorentz dispersions, and more general cases have been derived. It has been also shown that the PML for lossy medium can be easily derived from our PML as a special case. While our discussion was restricted to the frequency-dependent permittivities, the extension to the frequency-dependent magnetic materials is straightforward by using the same approach discussed in this paper.

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